An Introduction to the Theory of Point Processes: Volume I: Elementary Theory and Methods, Second Edition

D.J. Daley
D. Vere-Jones

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Volume I: Elementary Theory and Methods

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To Nola,
and in memory of Mary

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Preface to the Second Edition

In preparing this second edition, we have taken the opportunity to reshape the book, partly in response to the further explosion of material on point processes that has occurred in the last decade but partly also in the hope of making some of the material in later chapters of the first edition more accessible to readers primarily interested in models and applications. Topics such as conditional intensities and spatial processes, which appeared relatively advanced and technically difficult at the time of the first edition, have now been so extensively used and developed that they warrant inclusion in the earlier introductory part of the text. Although the original aim of the book—to present an introduction to the theory in as broad a manner as we are able—has remained unchanged, it now seems to us best accomplished in two volumes, the first concentrating on introductory material and models and the second on structure and general theory. The major revisions in this volume, as well as the main new material, are to be found in Chapters 6–8. The rest of the book has been revised to take these changes into account, to correct errors in the first edition, and to bring in a range of new ideas and examples.

Even at the time of the first edition, we were struggling to do justice to the variety of directions, applications and links with other material that the theory of point processes had acquired. The situation now is a great deal more daunting. The mathematical ideas, particularly the links to statistical mechanics and with regard to inference for point processes, have extended considerably. Simulation and related computational methods have developed even more rapidly, transforming the range and nature of the problems under active investigation and development. Applications to spatial point patterns, especially in connection with image analysis but also in many other scientific disciplines, have also exploded, frequently acquiring special language and techniques in the different fields of application. Marked point processes, which were clamouring for greater attention even at the time of the first edition, have acquired a central position in many of these new applications, influencing both the direction of growth and the centre of gravity of the theory.

We are sadly conscious of our inability to do justice to this wealth of new material. Even less than at the time of the first edition can the book claim to provide a comprehensive, up-to-the-minute treatment of the subject. Nor are we able to provide more than a sketch of how the ideas of the subject have evolved. Nevertheless, we hope that the attempt to provide an introduction to the main lines of development, backed by a succinct yet rigorous treatment of the theory, will prove of value to readers in both theoretical and applied fields and a possible starting point for the development of lecture courses on different facets of the subject. As with the first edition, we have endeavoured to make the material as self-contained as possible, with references to background mathematical concepts summarized in the appendices, which appear in this edition at the end of Volume I.

We would like to express our gratitude to the readers who drew our attention to some of the major errors and omissions of the first edition and will be glad to receive similar notice of those that remain or have been newly introduced. Space precludes our listing these many helpers, but we would like to acknowledge our indebtedness to Rick Schoenberg, Robin Milne, Volker Schmidt, Günter Last, Peter Glynn, Olav Kallenberg, Martin Kalinke, Jim Pitman, Tim Brown and Steve Evans for particular comments and careful reading of the original or revised texts (or both). Finally, it is a pleasure to thank John Kimmel of Springer-Verlag for his patience and encouragement, and especially Eileen Dallwitz for undertaking the painful task of rekeying the text of the first edition.

The support of our two universities has been as unflagging for this endeavour as for the first edition; we would add thanks to host institutions of visits to the Technical University of Munich (supported by a Humboldt Foundation Award), University College London (supported by a grant from the Engineering and Physical Sciences Research Council) and the Institute of Mathematics and its Applications at the University of Minnesota.

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Preface to the First Edition

This book has developed over many years—too many, as our colleagues and families would doubtless aver. It was conceived as a sequel to the review paper that we wrote for the Point Process Conference organized by Peter Lewis in 1971. Since that time the subject has kept running away from us faster than we could organize our attempts to set it down on paper. The last two decades have seen the rise and rapid development of martingale methods, the surge of interest in stochastic geometry following Rollo Davidson's work, and the forging of close links between point processes and equilibrium problems in statistical mechanics.

Our intention at the beginning was to write a text that would provide a survey of point process *theory* accessible to beginning graduate students and workers in applied fields. With this in mind we adopted a partly historical approach, starting with an informal introduction followed by a more detailed discussion of the most familiar and important examples, and then moving gradually into topics of increased abstraction and generality. This is still the basic pattern of the book. Chapters 1–4 provide historical background and treat fundamental special cases (Poisson processes, stationary processes on the line, and renewal processes). Chapter 5, on finite point processes, has a bridging character, while Chapters 6–14 develop aspects of the general theory.

The main difficulty we had with this approach was to decide when and how far to introduce the abstract concepts of functional analysis. With some regret, we finally decided that it was idle to pretend that a general treatment of point processes could be developed without this background, mainly because the problems of existence and convergence lead inexorably to the theory of measures on metric spaces. This being so, one might as well take advantage of the metric space framework from the outset and let the point process itself be defined on a space of this character: at least this obviates the tedium of having continually to specify the dimensions of the Euclidean space, while in the context of completely separable metric spaces—and this is the greatest generality we contemplate—intuitive spatial notions still provide a reasonable guide to basic properties. For these reasons the general results from Chapter 6 onward are couched in the language of this setting, although the examples continue to be drawn mainly from the one- or two-dimensional Euclidean spaces $\mathbb{R}^1$ and $\mathbb{R}^2$. Two appendices collect together the main results we need from measure theory and the theory of measures on metric spaces. We hope that their inclusion will help to make the book more readily usable by applied workers who wish to understand the main ideas of the general theory without themselves becoming experts in these fields. Chapter 13, on the martingale approach, is a special case. Here the context is again the real line, but we added a third appendix that attempts to summarize the main ideas needed from martingale theory and the general theory of processes. Such special treatment seems to us warranted by the exceptional importance of these ideas in handling the problems of inference for point processes.

In style, our guiding star has been the texts of Feller, however many light-years we may be from achieving that goal. In particular, we have tried to follow his format of motivating and illustrating the general theory with a range of examples, sometimes didactical in character, but more often taken from real applications of importance. In this sense we have tried to strike a mean between the rigorous, abstract treatments of texts such as those by Matthes, Kerstan and Mecke (1974/1978/1982) and Kallenberg (1975, 1983), and practically motivated but informal treatments such as Cox and Lewis (1966) and Cox and Isham (1980).

Numbering Conventions. Each chapter is divided into sections, with consecutive labelling within each of equations, statements (encompassing Definitions, Conditions, Lemmas, Propositions, Theorems), examples, and the exercises collected at the end of each section. Thus, in Section 1.2, (1.2.3) is the third equation, **Statement 1.2.III** is the third statement, EXAMPLE 1.2(c) is the third example, and Exercise 1.2.3 is the third exercise. The exercises are varied in both content and intention and form a significant part of the text. Usually, they indicate extensions or applications (or both) of the theory and examples developed in the main text, elaborated by hints or references intended to help the reader seeking to make use of them. The symbol □ denotes the end of a proof. Instead of a name index, the listed references carry page number(s) where they are cited. A general outline of the notation used has been included before the main text.

It remains to acknowledge our indebtedness to many persons and institutions. Any reader familiar with the development of point process theory over the last two decades will have no difficulty in appreciating our dependence on the fundamental monographs already noted by Matthes, Kerstan and Mecke in its three editions (our use of the abbreviation MKM for the 1978 English edition is as much a mark of respect as convenience) and Kallenberg in its two editions. We have been very conscious of their generous interest in our efforts from the outset and are grateful to Olav Kallenberg in particular for saving us from some major blunders. A number of other colleagues, notably

David Brillinger, David Cox, Klaus Krickeberg, Robin Milne, Dietrich Stoyan, Mark Westcott, and Deng Yonglu, have also provided valuable comments and advice for which we are very grateful. Our two universities have responded generously with seemingly unending streams of requests to visit one another at various stages during more intensive periods of writing the manuscript. We also note visits to the University of California at Berkeley, to the Center for Stochastic Processes at the University of North Carolina at Chapel Hill, and to Zhongshan University at Guangzhou. For secretarial assistance we wish to thank particularly Beryl Cranston, Sue Watson, June Wilson, Ann Milligan, and Shelley Carlyle for their excellent and painstaking typing of difficult manuscript.

Finally, we must acknowledge the long-enduring support of our families, and especially our wives, throughout: they are not alone in welcoming the speed and efficiency of Springer-Verlag in completing this project.

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10 Special Classes of Processes
11 Convergence Concepts and Limit Theorems
12 Ergodic Theory and Stationary Processes
13 Palm Theory
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15 Spatial Point Processes

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Principal Notation

Very little of the general notation used in Appendices 1–3 is given below. Also, notation that is largely confined to one or two sections of the same chapter is mostly excluded, so that neither all the symbols used nor all the uses of the symbols shown are given. The repeated use of some symbols occurs as a result of point process theory embracing a variety of topics from the theory of stochastic processes. Where they are given, page numbers indicate the first or significant use of the notation. Generally, the particular interpretation of symbols with more than one use is clear from the context.

Throughout the lists below, N denotes a point process and ξ denotes a random measure.

Spaces

$\mathbb{C}$	complex numbers	
$\mathbb{R}^d$	d-dimensional Euclidean space	
$\mathbb{R} = \mathbb{R}^1$	real line	
$\mathbb{R}_+$	nonnegative numbers	
$\mathbb{S}$	circle group and its representation as $(0, 2\pi]$	
$\mathbb{U}^d_{2\alpha}$	d-dimensional cube of side length 2α and vertices $(\pm\,\alpha, \ldots, \pm\,\alpha)$	
$\mathbb{Z},\ \mathbb{Z}_+$	integers of $\mathbb{R}$, $\mathbb{R}_+$	
$\mathcal{X}$	state space of N or ξ; often $\mathcal{X} = \mathbb{R}^d$; always $\mathcal{X}$ is c.s.m.s. (complete separable metric space)	
Ω	space of probability elements ω	
$\emptyset,\ \emptyset(\cdot)$	null set, null measure	
$\mathcal{E}$	measurable sets in probability space	
$(\Omega, \mathcal{E}, \mathcal{P})$	basic probability space on which N and ξ are defined	158
$\mathcal{X}^{(n)}$	n-fold product space $\mathcal{X} \times \cdots \times \mathcal{X}$	123
$\mathcal{X}^{\cup}$	$= \mathcal{X}^{(0)} \cup \mathcal{X}^{(1)} \cup \cdots$	129

$\mathcal{B}(\mathcal{X})$	Borel σ-field generated by open spheres of c.s.m.s. $\mathcal{X}$	34
$\mathcal{B}_{\mathcal{X}}$	$= \mathcal{B}(\mathcal{X}),\ \mathcal{B} = \mathcal{B}_{\mathbb{R}} = \mathcal{B}(\mathbb{R})$	34, 374
$\mathcal{B}_{\mathcal{X}}^{(n)} = \mathcal{B}(\mathcal{X}^{(n)})$	product σ-field on product space $\mathcal{X}^{(n)}$	129
$\mathrm{BM}(\mathcal{X})$	measurable functions of bounded support	161
$\mathrm{BM}_+(\mathcal{X})$	measurable nonnegative functions of bounded support	161
$\mathcal{K}$	mark space for marked point process (MPP)	194
$\mathcal{M}_{\mathcal{X}}\ (\mathcal{N}_{\mathcal{X}})$	totally finite (counting) measures on c.s.m.s. $\mathcal{X}$	158, 398
$\mathcal{M}_{\mathcal{X}}^{\#}$	boundedly finite measures on c.s.m.s. $\mathcal{X}$	158, 398
$\mathcal{N}_{\mathcal{X}}^{\#}$	boundedly finite counting measures on c.s.m.s. $\mathcal{X}$	131
$\mathcal{P}^+$	p.p.d. (positive positive-definite) measures	359
$\mathcal{S}$	infinitely differentiable functions of rapid decay	357
$\mathcal{U}$	complex-valued Borel measurable functions on $\mathcal{X}$ of modulus ≤ 1	144
$\mathcal{U} \otimes \mathcal{V}$	product topology on product space $\mathcal{X} \times \mathcal{Y}$ of topological spaces $(\mathcal{X},\mathcal{U})$, $(\mathcal{Y},\mathcal{V})$	378
$\mathcal{V} = \mathcal{V}(\mathcal{X})$	$[0,1]$-valued measurable functions $h(x)$ with $1 - h(x)$ of bounded support in $\mathcal{X}$	149, 152

General

Unless otherwise specified, $A \in \mathcal{B}_{\mathcal{X}}$, k and $n \in \mathbb{Z}_+$, t and $x \in \mathbb{R}$, $h \in \mathcal{V}(\mathcal{X})$, and $z \in \mathbb{C}$.

$\tilde{}$	$\tilde{\nu}, \widetilde{F}$ = Fourier–Stieltjes transforms of measure ν or d.f. F	411–412
	$\tilde{\phi}$ = Fourier transform of Lebesgue integrable function ϕ for counting measures	357
$\breve{}$	reduced (ordinary or factorial) (moment or cumulant) measure	160
#	extension of concept from totally finite to boundedly finite measure space	158
$\Vert\mu\Vert$	variation norm of measure μ	374
a.e. μ, μ-a.e.	almost everywhere with respect to measure μ	376
a.s., $\mathcal{P}$-a.s.	almost sure, $\mathcal{P}$-almost surely	376
$A^{(n)}$	n-fold product set $A \times \cdots \times A$	130
$\mathcal{A}$	family of sets generating $\mathcal{B}$; semiring of bounded Borel sets generating $\mathcal{B}_{\mathcal{X}}$	31, 368
$B_u\ (T_u)$	backward (forward) recurrence time at u	58, 76
$c_k,\ c_{[k]}$	kth cumulant, kth factorial cumulant, of distribution $\{p_n\}$	116
$c(x) = c(y, y + x)$	covariance density of stationary mean square continuous process on $\mathbb{R}^d$	160, 358

$C_{[k]}(\cdot),\ c_{[k]}(\cdot)$	factorial cumulant measure and density	147
$\breve{C}_2(\cdot), \breve{c}(\cdot)$	reduced covariance measure of stationary N or ξ	292
$\breve{c}(\cdot)$	reduced covariance density of stationary N or ξ	160, 292
$\delta(\cdot)$	Dirac delta function	
$\delta_x(A)$	Dirac measure, $= \int_A \delta(u-x)\,\mathrm{d}u = I_A(x)$	382
$\Delta F(x) = F(x) - F(x-)$		
	jump at x in right-continuous function F	107
$\mathrm{e}_\lambda(x) = (\frac{1}{2}\lambda)^d \exp\bigl(-\lambda \sum_{i=1}^d \lvert x_i \rvert\bigr)$		
	two-sided exponential density in $\mathbb{R}^d$	359
F	renewal process lifetime d.f.	67
F^{n*}	n-fold convolution power of measure or d.f. F	55
$F(\cdot\,;\cdot)$	finite-dimensional (fidi) distribution	158–161
$\boldsymbol{\mathcal{F}}$	history	236, 240
$\Phi(\cdot)$	characteristic functional	15
$G[h]$	probability generating functional (p.g.fl.) of N,	15, 144
$G[h \mid x]$	member of measurable family of p.g.fl.s	166
$G_c[\cdot],\ G_m[\cdot \mid x]$	p.g.fl.s of cluster centre and cluster member processes N_c and $N_m(\cdot \mid x)$	178
$G,\ G_I$	expected information gain (per interval) of stationary N on $\mathbb{R}$	280, 285
$\Gamma(\cdot),\ \gamma(\cdot)$	Bartlett spectrum, its density when it exists	304
$H(\mathcal{P};\mu)$	generalized entropy	277, 283
$\boldsymbol{\mathcal{H}},\ \boldsymbol{\mathcal{H}}^*$	internal history of ξ on $\mathbb{R}_+$, $\mathbb{R}$	236
$I_A(x) = \delta_x(A)$	indicator function of element x in set A	
$I_n(x)$	modified Bessel function of order n	72
$J_n(A_1 \times \cdots \times A_n)$		
	Janossy measure	124
$j_n(x_1, \ldots, x_n)$	Janossy density	125
$J_n(\cdot \mid A)$	local Janossy measure	137
K	compact set	371
$K_n(\cdot),\ k_n(\cdot)$	Khinchin measure and density	146
$\ell(\cdot)$	Lebesgue measure in $\mathcal{B}(\mathbb{R}^d)$,	31
	Haar measure on σ-group	408–409
$L_u = B_u + T_u$	current lifetime of point process on $\mathbb{R}$	58, 76
$L[f]\ (f \in BM_+(\mathcal{X}))$		
	Laplace functional of ξ	161
$L_\xi[1-h]$	p.g.fl. of Cox process directed by ξ	170
$L_2(\xi^0),\ L_2(\Gamma)$	Hilbert spaces of square integrable r.v.s ξ^0, and of functions square integrable w.r.t. measure Γ	332
$L_A(x_1, \ldots, x_n), = j_N(x_1, \ldots, x_N \mid A)$		
	likelihood, local Janossy density, $N \equiv N(A)$	22, 212
λ	rate of N, especially intensity of stationary N	46
$\lambda^*(t)$	conditional intensity function	231
$m_k\ (m_{[k]})$	kth (factorial) moment of distribution $\{p_n\}$	115

$\breve{m}_2$, $\breve{M}_2$	reduced second-order moment density, measure, of stationary N	289
m_{g}	mean density of ground process N_{g} of MPP N	198, 323
$N(A)$	number of points in A	42
$N(a,b]$	number of points in half-open interval $(a,b]$,	19
	$= N((a,b])$	42
$N(t)$	$= N(0,t] = N((0,t])$	42
N_c	cluster centre process	176
$N(\cdot \mid x)$	cluster member or component process	176
$\{(p_n, \Pi_n)\}$	elements of probability measure for finite point process	123
$P(z)$	probability generating function (p.g.f.) of distribution $\{p_n\}$	10, 115
$P(x,A)$	Markov transition kernel	92
$P_0(A)$	avoidance function	31, 135
$\mathcal{P}_{jk}$	set of j-partitions of $\{1,\ldots,k\}$	121
$\mathcal{P}$	probability measure of stationary N on $\mathbb{R}$,	53
	probability measure of N or ξ on c.s.m.s. $\mathcal{X}$	158
$\{\pi_k\}$	batch-size distribution	28, 51
$q(x) = f(x)/[1 - F(x)]$	hazard function for lifetime d.f. F	2, 106
$Q(z)$	$= -\log P(z)$	27
$Q(\cdot)$, $Q(t)$	hazard measure, integrated hazard function (IHF)	109
$\rho(x,y)$	metric for x, y in metric space	370
$\{S_n\}$	random walk, sequence of partial sums	66
$S(x) = 1 - F(x)$	survivor function of d.f. F	2, 109
$S_r(x)$	sphere of radius r, centre x, in metric space $\mathcal{X}$	35, 371
$t(x) = \prod_{i=1}^{d}(1 - \lvert x_i \rvert)_+$	triangular density in $\mathbb{R}^d$	359
T_u	forward recurrence time at u	58, 75
$\mathcal{T} = \{S_1(\mathcal{T}),\ldots,S_j(\mathcal{T})\}$	a j-partition of k	121
$\mathcal{T} = \{\mathcal{T}_n\} = \{\{A_{ni}\}\}$	dissecting system of nested partitions	382
$U(A) = \mathrm{E}[N(A)]$	renewal measure	67
$U(x)$	$= U([0,x])$, expectation function,	61
	renewal function ($U(x) = 1 + U_0(x)$)	67
$V(A)$	$= \operatorname{var} N(A)$, variance function	295
$V(x) = V((0,x])$	variance function for stationary N or ξ on $\mathbb{R}$	80, 301
$\{X_n\}$	components of random walk $\{S_n\}$,	66
	intervals of Wold process	92